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ADDENDUM

Geometrical aspects in Yang–Mills gauge theories

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Abstract
A gauge-independent formulation of the theory developed in *J. Phys. A: Math. Gen.* **36** 8341 (2003) is proposed.

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1. Introduction

In a previous paper [1], we have developed a new geometrical framework for Yang–Mills theories on a principal fibre bundle $P \to M$, as an appropriate quotient space of the standard first jet-bundle.

The approach proposed in [1] allows us to take care from the beginning only of the physical degrees of freedom of the theory.

First, we have investigated the geometrical properties of the resulting mathematical setting, so extending some fundamental geometrical structures and constructions of standard jet-bundle geometry to the newly defined space.

Then, we have deduced the field equations from a variational problem formulated through a *regular* Poincaré–Cartan form, so letting the variational principle itself ensure the kinematical admissibility of critical sections.

Finally we have investigated the relationships between symmetries and conserved quantities in the newer scheme, stating a generalized Noether theorem.

The whole construction proposed in [1] has been developed by assuming a trivialization of P is fixed, once and for all, thus focusing our attention on a local analysis. By doing so, we have dealt with Yang–Mills fields and not with principal connection 1-forms. Actually, this way of proceeding is standard enough when one works on flat space–times or with trivial principal bundles.

Nevertheless, as underlined in a referee report on [1], a global gauge-independent formulation of the proposed theory would be more appropriate and, from a geometrical point of view, more elegant too. This is the aim of the present work.

In this addendum, we show in fact that the whole geometrical machinery developed in [1] admits a genuinely gauge-invariant formulation.

We start by considering the first jet-bundle $J_1(J_1P/G)$ of the bundle of principal connections $J_1P/G \to M$, G denoting the structural group of P. Following the main idea of our previous work, we quotient $J_1(J_1P/G)$ with respect to a suitable equivalence relation and extend all the results stated in [1] to the corresponding quotient space.

Also, we present a *Hamiltonian* description of the field equations, which could lay the grounds for further developments (at present under investigation) in the direction of *dual Lagrangian field theories* [2, 3].

2. The geometrical framework

Let $\pi : P \to M$ be a principal fibre bundle, with structural group G, referred to local fibred coordinates $x^i, g^\mu, i = 1, \ldots, m = \dim M, \mu = 1, \ldots, r = \dim G$. Changes of trivialization of P give rise to coordinate transformations in P of the form

$$\bar{x}^i = \bar{x}^i(x^j) \qquad \bar{g}^\mu = (\gamma^{-1}(x) \cdot g)^\mu \tag{2.1}$$

where $\gamma : U \subset M \to G$ (U an open set) are arbitrary smooth maps, and $\gamma^{-1}(x) \cdot g$ indicates the product between γ^{-1} and g in G.

Let J_1P denote the first jet-bundle of $\pi : P \to M$, referred to local jet-coordinates $x^i, g^\mu, g^\mu_i \left(\simeq \frac{\partial g^\mu}{\partial x^i}\right)$.

As is well known, the space of principal connections on P may be identified with the quotient space J_1P/G, the quotient being performed with respect to the action generated by the first jet-prolongations J_1R_r of the right maps $R_r, \forall\, r \in G$.

In more detail, the action of the maps J_1R_r on J_1P is given by $\left(x^i, g^\mu, g^\mu_i\right) \to (x^i, (gr)^\mu, g^\nu_i V^\mu_\nu(g, r), V^\mu_\nu$ denoting the differential of the right multiplication in G.[3] Local coordinates on the quotient space J_1P/G are $x^i, v^\mu_i := g^\nu_i V^\mu_\nu(g, g^{-1})$.

The manifold J_1P/G is an affine bundle over M and principal connections on P may be viewed as sections $\omega : M \to J_1P/G$. We may put on J_1P/G local coordinates $x^i, a^\mu_i := -v^\mu_i$ in such a way that every section $\omega : x \to \left(x, a^\mu_i(x)\right)$ yields the corresponding principal connection 1-form (still denoted by ω) on P

$$\omega(x, g) = \omega^\mu(x, g) \otimes \underline{e}_\mu := \left[Ad(g^{-1})^\mu_\nu a^\nu_i(x)\, dx^i + W^\mu_\nu(g^{-1}, g)\, dg^\nu\right] \otimes \underline{e}_\mu \tag{2.2}$$

where Ad^μ_ν and W^μ_ν denote respectively the adjoint representation of G and the differential of the left multiplication in G, while $\underline{e}_\mu (\mu = 1, \ldots, r)$ indicate a basis of the Lie algebra $\mathfrak{g}$ of G.

It is worth noting that changes of coordinates (2.1) in P induce coordinate transformations in J_1P/G expressed as

$$\bar{x}^i = \bar{x}^i(x^j) \qquad \bar{a}^\mu_i = \left[Ad(\gamma^{-1})^\mu_\nu a^\nu_j + W^\mu_\nu(\gamma^{-1}, \gamma)\frac{\partial \gamma^\nu}{\partial x^j}\right]\frac{\partial x^j}{\partial \bar{x}^i}. \tag{2.3}$$

Now, let $\hat{\pi} : J_1(J_1P/G) \to M$ be the first jet-bundle associated with the bundle $J_1P/G \to M$. Local coordinates on $J_1(J_1P/G)$ are $x^i, a^\mu_i, a^\mu_{ij} \left(\simeq \frac{\partial a^\mu_i}{\partial x^j}\right)$ undergoing transformation laws (2.3) together with

$$\bar{a}^\mu_{ik} = \frac{\partial x^j}{\partial \bar{x}^i}\frac{\partial x^h}{\partial \bar{x}^k}\left[Ad(\gamma^{-1})^\mu_\nu a^\nu_{jh} + \frac{\partial Ad(\gamma^{-1})^\mu_\nu}{\partial x^h} a^\nu_j + \frac{\partial \eta^\mu_j}{\partial x^h}\right] + \frac{\partial^2 x^j}{\partial \bar{x}^k \partial \bar{x}^i}\left[Ad(\gamma^{-1})^\mu_\nu a^\nu_j + \eta^\mu_j\right] \tag{2.4}$$

where $\eta^\mu_j(x) := W^\mu_\nu(\gamma^{-1}(x), \gamma(x))\frac{\partial \gamma^\nu(x)}{\partial x^j}$.

[3] More explicitly, if we denote by $g = x \cdot y$ the product in G, we set $dg^\mu = dx^\lambda V^\mu_\lambda + dy^\lambda W^\mu_\lambda$.

Borrowing from [1], we define in $J_1(J_1P/G)$ the following equivalence relation: given two points $w_1 = \left(x^i, a_i^\mu, a_{ij}^\mu\right)$ and $w_2 = \left(x^i, \hat{a}_i^\mu, \hat{a}_{ij}^\mu\right) \in J_1(J_1P/G)$ with $\hat{\pi}(w_1) = \hat{\pi}(w_2) \in M$, then $w_1 \sim w_2 \Leftrightarrow a_i^\mu = \hat{a}_i^\mu$ and $\left(a_{ij}^\mu - a_{ji}^\mu\right) = \left(\hat{a}_{ij}^\mu - \hat{a}_{ji}^\mu\right)$. Making use of transformation laws (2.4), it is a straightforward matter to check that this equivalence relation is well defined since it is independent of the choice of local coordinates.

From a geometrical viewpoint, taking the very definition of $J_1(J_1P/G)$ as well as the representation (2.2) into account, it is easily seen that the above-introduced equivalence relation amounts to stating two principal connection 1-forms ω_1 and ω_2 are *equivalent* if they have a first-order contact with respect to the exterior differentiation[4]. More explicitly, if ω_1 and ω_2 denote two principal connection 1-forms on P (i.e. two sections of $J_1P/G \to M$) representing two points $w_1, w_2 \in J_1(J_1P/G)$ (with $x = \hat{\pi}(w_1) = \hat{\pi}(w_2) \in M$) respectively, then we say that

$$w_1 \sim w_2 \Leftrightarrow \omega_1(p) = \omega_2(p) \qquad \text{and} \qquad \mathrm{d}\omega_1(p) = \mathrm{d}\omega_2(p) \qquad \forall p \in \pi^{-1}(x) \subset P.$$

We wish to stress that this equivalence relation is related to the exterior derivative and not to the standard first-order contact condition, usually encountered in the definition of first jet-bundles.

We denote by $\mathcal{J}(P)$ the quotient space $J_1(J_1P/G)/\sim$ and by $\rho : J_1(J_1P/G) \to \mathcal{J}(P)$ the canonical projection. Also, we put on $\mathcal{J}(P)$ local coordinates $x^i, a_i^\mu, A_{ij}^\mu := \frac{1}{2}\left(a_{ij}^\mu - a_{ji}^\mu\right)$ $(i < j)$, subject to the transformation laws (2.3) and

$$\bar{A}_{ik}^\mu = \frac{\partial x^j}{\partial \bar{x}^i}\frac{\partial x^h}{\partial \bar{x}^k}\left[Ad(\gamma^{-1})_\nu^\mu A_{jh}^\nu + \frac{1}{2}\left(\frac{\partial Ad(\gamma^{-1})_\nu^\mu}{\partial x^h}a_j^\nu - \frac{\partial Ad(\gamma^{-1})_\nu^\mu}{\partial x^j}a_h^\nu\right)\right.$$
$$\left. + \frac{1}{2}\left(\frac{\partial \eta_j^\mu}{\partial x^h} - \frac{\partial \eta_h^\mu}{\partial x^j}\right)\right]. \tag{2.5}$$

As in [1], we may adapt some standard geometrical jet-structures, such as jet-extensions, contact forms and jet-prolongations, to the newly defined space $\mathcal{J}(P)$. In detail we have:

2.1. $\mathcal{J}$-extension of sections

Given a section $\sigma : M \to J_1P/G$, we define its $\mathcal{J}$-extension $\mathcal{J}\sigma : M \to \mathcal{J}(P)$ as $\mathcal{J}\sigma := \rho \circ j_1\sigma$, $j_1\sigma : M \to J_1(J_1P/G)$ indicating the standard first jet-extension of σ.

Any section $\gamma : M \to \mathcal{J}(P)$ is said to be *holonomic* if there exists a section $\sigma : M \to J_1P/G$ such that $\gamma = \mathcal{J}\sigma$. Every holonomic section γ is then expressed locally as $\gamma : x \to \left(x^i, a_i^\mu(x), A_{ij}^\mu(x) = \frac{1}{2}\left(\frac{\partial a_i^\mu(x)}{\partial x^j} - \frac{\partial a_j^\mu(x)}{\partial x^i}\right)\right)$.

2.2. Contact forms

Let us define on $\mathcal{J}(P)$ the following 2-forms,

$$\theta^\mu := \mathrm{d}a_j^\mu \wedge \mathrm{d}x^j + A_{ij}^\mu \,\mathrm{d}x^i \wedge \mathrm{d}x^j \tag{2.6}$$

where we have used the notation $A_{ij}^\mu = -A_{ji}^\mu$ (henceforth systematically adopted) whenever $i > j$.

Under changes of coordinates (2.3) and (2.5), we have the transformation laws

$$\bar{\theta}^\mu = Ad(\gamma^{-1})_\nu^\mu \theta^\nu$$

thus ensuring the invariance of the module generated locally by the 2-forms (2.6).

The bundle spanned locally by the forms (2.6) is called the *contact bundle* over $\mathcal{J}(P)$ and it is denoted by $C(\mathcal{J}(P))$; sections $\eta : \mathcal{J}(P) \to C(\mathcal{J}(P))$ are called contact 2-forms on $\mathcal{J}(P)$.

[4] Equivalently, one could use the covariant exterior differentiation D instead of the exterior differentiation d.

Moreover, it is easily seen that a section $\gamma : M \to \mathcal{J}(P)$ is holonomic if and only if $\gamma^*(\eta) = 0$ for every contact form η.

2.3. $\mathcal{J}$-prolongation of morphisms

Following [1], we characterize those bundle morphisms of J_1P/G

$$\begin{array}{ccc} J_1P/G & \xrightarrow{\Phi} & J_1P/G \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi} \\ M & \xrightarrow{\chi} & M \end{array}$$

projecting to diffeomorphisms of M, whose ordinary jet-prolongations $j_1\Phi$ on $J_1(J_1P/G)$ satisfy the requirement

$$\rho \circ j_1\Phi(w_1) = \rho \circ j_1\Phi(w_2) \qquad \forall\, w_1, w_2 \in \rho^{-1}(z) \tag{2.7}$$

for any $z \in \mathcal{J}(P)$. For every bundle morphism (Φ, χ) obeying ansatz (2.7), there is a well-defined associated map $\mathcal{J}\Phi : \mathcal{J}(P) \to \mathcal{J}(P)$ expressed as

$$\mathcal{J}\Phi(z) := \rho \circ j_1\Phi(w) \qquad \forall\, w \in \rho^{-1}(z) \quad z \in \mathcal{J}(E)$$

henceforth referred to as the $\mathcal{J}$-prolongation of (Φ, χ).

By repeating the same arguments as in [1], we conclude that the most general bundle morphism (Φ, χ) satisfying (2.7) is locally represented in the form

$$\begin{cases} y^i = \chi^i(x^j) \\ b_i^\nu = \Phi_i^\nu\big(x^j, a_j^\mu\big) = \Gamma_\mu^\nu(x)\dfrac{\partial x^r}{\partial y^i}a_r^\mu + f_i^\nu(x) \end{cases} \tag{2.8}$$

where $\Gamma_\mu^\nu(x)$ and $f_i^\nu(x)$ are arbitrary functions on M. The only difference with respect to [1] is the need to check that the formal expressions (2.8) are invariant under changes of coordinates (2.3). In this connection, after a direct calculation, we end up with the transformation laws

$$\begin{cases} \bar{y}^i = \bar{\chi}^i(\bar{x}^j) \\ \bar{b}_i^\nu = \bar{\Gamma}_\mu^\nu(\bar{x})\dfrac{\partial \bar{x}^r}{\partial \bar{y}^i}\bar{a}_r^\mu + \bar{f}_i^\nu(\bar{x}) \end{cases}$$

where

$$\bar{\Gamma}_\mu^\nu(\bar{x}) := \Gamma^\lambda_{\eta\,|x(\bar{x})} Ad(\gamma^{-1})^\nu_{\lambda|\chi(x(\bar{x}))} Ad(\gamma)^\eta_{\mu|\bar{x}}$$

and

$$\bar{f}_j^\nu(\bar{x}) := \left[\Gamma^\lambda_{\rho|x(\bar{x})} \frac{\partial \bar{y}^s}{\partial \bar{x}^j}_{|\bar{\chi}(\bar{x})} \bar{\eta}_s^\rho + \frac{\partial x^i}{\partial \bar{x}^j} f_i^\lambda{}_{|x(\bar{x})}\right] Ad(\gamma^{-1})^\nu_{\lambda|\chi(x(\bar{x}))} + \frac{\partial x^i}{\partial \bar{x}^j}\eta_i^\nu{}_{|\chi(x(\bar{x}))}$$

thus proving the required invariance.

In local coordinates, the explicit expression of the $\mathcal{J}$-prolongation $\mathcal{J}\Phi$ of a $\mathcal{J}$-prolongable bundle morphism (2.8) is given by [1]

$$\begin{cases} y^i = \chi^i(x^k) \\ b_i^\nu = \Gamma_\mu^\nu(x)\frac{\partial x^r}{\partial y^i}a_r^\mu + f_i^\nu(x) \\ B_{ij}^\nu = \Gamma_\mu^\nu A_{ks}^\mu \dfrac{\partial x^k}{\partial y^i}\dfrac{\partial x^s}{\partial y^j} + \dfrac{1}{2}\left[\dfrac{\partial \Gamma_\mu^\nu}{\partial x^k}\left(\dfrac{\partial x^k}{\partial y^j}\dfrac{\partial x^r}{\partial y^i} - \dfrac{\partial x^k}{\partial y^i}\dfrac{\partial x^r}{\partial y^j}\right)a_r^\mu + \dfrac{\partial f_i^\nu}{\partial x^k}\dfrac{\partial x^k}{\partial y^j} - \dfrac{\partial f_j^\nu}{\partial x^k}\dfrac{\partial x^k}{\partial y^i}\right]. \end{cases}$$

As happens for standard jet-prolongations in ordinary jet-bundles [4], $\mathcal{J}$-prolongations of bundle morphisms are characterized by preserving contact forms and $\mathcal{J}$-extensions (see propositions 2.2 and 2.3 in [1]).

2.4. $\mathcal{J}$-prolongation of vector fields

Let us characterize those vector fields X on J_1P/G, projecting to M, whose first jet-prolongations $J_1(X)$ on $J_1(J_1P/G)$ pass to the quotient $\mathcal{J}(P)$.

Once again, by repeating some considerations, analogous to those made in [1], one can prove that such vector fields are of the only form

$$X = \epsilon^i(x^j)\frac{\partial}{\partial x^i} + \left(-\frac{\partial \epsilon^k}{\partial x^q}a^\mu_k + D^\mu_\nu(x^j)a^\nu_q + G^\mu_q(x^j)\right)\frac{\partial}{\partial a^\mu_q} \tag{2.9}$$

where $\epsilon^i(x)$, $D^\mu_\nu(x)$ and $G^\mu_q(x)$ are arbitrary functions on M. As above, we have to verify that the representations (2.9) are invariant under changes of coordinates (2.3). In this respect, straightforward calculations show that the vector fields (2.9) undergo the transformation law

$$X = \bar{\epsilon}^i\frac{\partial}{\partial \bar{x}^i} + \left(-\frac{\partial \bar{\epsilon}^k}{\partial \bar{x}^q}\bar{a}^\mu_k + \bar{D}^\mu_\nu \bar{a}^\nu_q + \bar{G}^\mu_q\right)\frac{\partial}{\partial \bar{a}^\mu_q}$$

where

$$\bar{\epsilon}^i := \epsilon^k\frac{\partial \bar{x}^i}{\partial x^k}$$

$$\bar{D}^\mu_\nu := D^\eta_\rho Ad(\gamma)^\rho_\nu Ad(\gamma^{-1})^\mu_\eta + Ad(\gamma)^\eta_\nu \epsilon^i \frac{\partial Ad(\gamma^{-1})^\mu_\eta}{\partial x^i}$$

and

$$\bar{G}^\mu_q := \epsilon^i\frac{\partial}{\partial x^i}\left(\frac{\partial x^k}{\partial \bar{x}^q}\eta^\mu_k\right) + G^\nu_j\frac{\partial x^j}{\partial \bar{x}^q}Ad(\gamma^{-1})^\mu_\nu$$

thus proving the required result.

For each vector field (2.9), its $\mathcal{J}$-prolongation $\mathcal{J}(X) : \mathcal{J}(P) \to T\mathcal{J}(P)$ is then well defined as

$$\mathcal{J}(X)(z) := \rho_{*\rho^{-1}(z)}(j_1(X)) \qquad \forall\, z \in \mathcal{J}(P) \tag{2.10}$$

amounting to taking the standard first jet-prolongation $J_1(X)$ and projecting it on $\mathcal{J}(P)$. In local coordinates we have [1]

$$\mathcal{J}(X) = \epsilon^i(x^j)\frac{\partial}{\partial x^i} + \left(-\frac{\partial \epsilon^k}{\partial x^q}a^\mu_k + D^\mu_\nu(x^j)a^\nu_q + G^\mu_q(x^j)\right)\frac{\partial}{\partial a^\mu_q} + \sum_{i<j} h^\mu_{ij}\frac{\partial}{\partial A^\mu_{ij}}$$

where

$$h^\mu_{ij} = \frac{1}{2}\left(\frac{\partial D^\mu_\nu}{\partial x^j}a^\nu_i - \frac{\partial D^\mu_\nu}{\partial x^i}a^\nu_j + \frac{\partial G^\mu_i}{\partial x^j} - \frac{\partial G^\mu_j}{\partial x^i}\right) + D^\mu_\nu A^\nu_{ij} + \left(A^\mu_{ki}\frac{\partial \epsilon^k}{\partial x^j} - A^\mu_{kj}\frac{\partial \epsilon^k}{\partial x^i}\right).$$

According to ordinary jet-prolongations [4], $\mathcal{J}$-prolongations (2.10) are characterized by preserving contact forms and are a Lie algebra (see proposition 2.4 and corollary 2.1 in [1]).

3. The field equations

Still following [1], in order to implement the field equations in the present framework, it is convenient to introduce new coordinates on $\mathcal{J}(P)$ of the form

$$x^i = x^i \qquad a^\mu_i = a^\mu_i \qquad F^\mu_{ji} = -2A^\mu_{ij} - a^\nu_i a^\rho_j C^\mu_{\rho\nu} \tag{3.1}$$

$C^\mu_{\rho\nu}$ being the structure coefficients of the Lie algebra $\mathfrak{g}$.

The idea is to take the components of the curvature tensor F^{μ}_{ji} as $\mathcal{J}$-coordinates. From equations (2.3) and (2.5) one can easily deduce the well-known gauge transformation law

$$\bar{F}^{\mu}_{ji} = Ad(\gamma^{-1})^{\mu}_{\nu} F^{\nu}_{pq} \frac{\partial x^p}{\partial \bar{x}^j} \frac{\partial x^q}{\partial \bar{x}^i}$$

for the curvature tensor F^{μ}_{ji}.

In this addendum, we consider Yang–Mills Lagrangians which can be expressed as

$$L = \mathcal{L}\big(x^i, a^{\mu}_i, F^{\mu}_{ij}\big)\,\mathrm{d}s \tag{3.2}$$

with $\mathrm{d}s := \mathrm{d}x^1 \wedge \ldots \wedge \mathrm{d}x^m$, and which are *regular*, namely which verify the condition $\det\Big\| \frac{\partial^2 \mathcal{L}}{\partial F^{\mu}_{ij} \partial F^{\sigma}_{pq}} \Big\| \neq 0$. We note that the standard pure Yang–Mills Lagrangian density $\mathcal{L} = -\frac{1}{4} F^{\mu}_{ij} F^{ij}_{\mu} \sqrt{-g}$ satisfies this requirement, being now regular in view of the definition of the new bundle $\mathcal{J}(P)$.

There is a corresponding Poincaré–Cartan m-form on $\mathcal{J}(P)$, associated with every Lagrangian (3.2), expressed as [1]

$$\Theta_L = \mathcal{L}\,\mathrm{d}s + \tfrac{1}{2}\theta^{\mu} \wedge P_{\mu} \tag{3.3}$$

where $P_{\mu} := \frac{\partial \mathcal{L}}{\partial F^{\mu}_{ij}}\,\mathrm{d}s_{ij}$, $\mathrm{d}s_{ij} := \frac{\partial}{\partial x^i} \rfloor \frac{\partial}{\partial x^j} \rfloor \mathrm{d}s$.

The field equations are deducible from a variational problem built on $\mathcal{J}(P)$ through the m-form (3.3). More precisely, they are seen to be the Euler–Lagrange equations associated with the functional

$$A_L(\gamma) := \int_D \gamma^*(\Theta_L) \qquad \forall \text{ section } \gamma : D \subset M \to \mathcal{J}(P),\ D \text{ compact domain.}$$

In fact, the requirement of stationarity for the functional A_L, together with the usual boundary conditions used in variational calculus, turn out to be mathematically equivalent to the condition

$$\gamma^*(X \rfloor \mathrm{d}\Theta_L) = 0 \qquad \forall\, X \in D^1(\mathcal{J}(P)). \tag{3.4}$$

in turn splitting into two sets of final equations, expressed respectively as

$$\gamma^*(\theta^{\mu}) = 0 \qquad \forall\, \mu = 1, \ldots, r \tag{3.5a}$$

and

$$\gamma^* \left(\frac{\partial \mathcal{L}}{\partial a^{\mu}_i} + D_j \frac{\partial \mathcal{L}}{\partial F^{\mu}_{ji}} \right) = 0. \tag{3.5b}$$

The first ones ensure the holonomy of the critical sections γ,[5] while the second ones represent the actual field equations of the problem.

In connection with this, we conclude this addendum by briefly proposing the 'Hamiltonian' counterpart of the above-outlined construction.

To start with, borrowing from [2] for notation, let $P \times_{Ad^*} \mathfrak{g}^*$ be the bundle associated with P through the co-adjoint action Ad^* of G on its dual Lie algebra $\mathfrak{g}^*$, and let $\bar{\Pi} := (P \times_{Ad^*} \mathfrak{g}^*) \otimes_M \Lambda^{m-2}(M)$ denote the tensor product over M between $P \times_{Ad^*} \mathfrak{g}^*$ and the space of the $(m-2)$-forms on M. Also, let us consider the fibred product over M $\Pi(P) := J_1 P/G \times_M \bar{\Pi}$, referred to local coordinates $x^i, a^{\mu}_i, P^{ij}_{\mu}$.

Then, the bundle $\Pi(P)$ identifies in a natural way with the *phase space* of the theory. To see this point, let us introduce the *Legendre map* from $\mathcal{J}(P)$ to $\Pi(P)$ by setting

$$P^{ij}_{\mu} := \frac{\partial \mathcal{L}}{\partial F^{\mu}_{ij}}. \tag{3.6}$$

[5] We stress that here the kinematic admissibility (3.5*a*) is obtained directly from the variational principle itself and it is not an *a priori* imposed condition. This is due to the fact that in the present framework the Yang–Mills Lagrangians become *regular* [1].

Under the regularity condition $\det \left\| \frac{\partial^2 \mathcal{L}}{\partial F^\mu_{ij} \partial F^\sigma_{pq}} \right\| \neq 0$, equations (3.6) define a (local) diffeomorphism ϕ_L, fibred over J_1P/G, according to the diagram

$$\begin{array}{ccc} \mathcal{J}(P) & \xrightarrow{\phi_L} & \Pi(P) \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi} \\ J_1P/G & = & J_1P/G. \end{array}$$

When the map ϕ_L is a global diffeomorphism, the Lagrangian L is said to be *hyper regular*. In this case, through a pull-back procedure, we may endow the manifold $\Pi(P)$ with the *Hamiltonian m*-form[6]

$$\Theta_H := \left(\phi_L^{-1}\right)^* \Theta_L. \tag{3.7}$$

In this connection, we recall that the Poincaré–Cartan form Θ_L may also be represented as (see [1] for more details)

$$\Theta_L = \mathcal{L}\, \mathrm{d}s - \frac{1}{2} \frac{\partial \mathcal{L}}{\partial F^\mu_{ji}} F^\mu_{ji}\, \mathrm{d}s - \frac{\partial \mathcal{L}}{\partial F^\mu_{ji}} \Omega^\mu_i \wedge \mathrm{d}s_j$$

where $\Omega^\mu_i := \mathrm{d}a^\mu_i + \frac{1}{2} a^\nu_i C^\mu_{\rho\nu} a^\rho_j\, \mathrm{d}x^j$.

From this, it is easily seen that equations (3.6) and (3.7) lead to the explicit local expression

$$\Theta_H = -\mathcal{H}\, \mathrm{d}s - P^{ji}_\mu \Omega^\mu_i \wedge \mathrm{d}s_j \tag{3.8}$$

$\mathcal{H}(x, a, P) := -\mathcal{L}(x, a, F(x, a, P)) + \frac{1}{2} P^{ji}_\mu F^\mu_{ji}(x, a, P)$ representing the *Hamiltonian density*.

As happens in the Lagrangian setting $\mathcal{J}(P)$, the field equations may be deduced from a variational principle on $\Pi(P)$, consisting in the study of the stationarity points for the functional

$$A_H(\gamma) := \int_D \gamma^*(\Theta_H) \qquad \forall \text{ section } \gamma : D \subset M \to \Pi(P),\ D \text{ compact domain.}$$

With standard boundary conditions, the ansatz $\delta A_H = 0$ amounts to the equation

$$\gamma^*(X \rfloor \mathrm{d}\Theta_H) = 0 \qquad \forall\, X \in D^1(\Pi(P)). \tag{3.9}$$

Taking vertical infinitesimal deformations $X = X^\mu_i \frac{\partial}{\partial a^\mu_i} + \sum_{i<j} X^{ij}_\mu \frac{\partial}{\partial P^{ij}_\mu}$ only into account for simplicity, we get

$$\begin{aligned} \gamma^*(X \rfloor \mathrm{d}\Theta_H) &= \gamma^* \left(-\frac{\partial \mathcal{H}}{\partial a^\mu_i} \mathrm{d}s - P^{ji}_\sigma C^\sigma_{\rho\mu} a^\rho_j\, \mathrm{d}s + \mathrm{d}P^{ji}_\mu \wedge \mathrm{d}s_j \right) X^\mu_{i\ |\gamma(x)} \\ &\quad + \gamma^* \left(-\frac{1}{2} \frac{\partial \mathcal{H}}{\partial P^{ij}_\mu} \mathrm{d}s - \mathrm{d}a^\mu_j \wedge \mathrm{d}s_j - \frac{1}{2} a^\nu_j C^\mu_{\rho\nu} a^\rho_i\, \mathrm{d}s \right) X^{ij}_{\mu\ |\gamma(x)}. \end{aligned}$$

In view of the arbitrariness of X, the last equation splits into

$$\frac{\partial \mathcal{H}}{\partial P^{ij}_\mu} = \frac{\partial a^\mu_i}{\partial x^j} - \frac{\partial a^\mu_j}{\partial x^i} - a^\nu_j C^\mu_{\rho\nu} a^\rho_i \tag{3.10a}$$

$$-\frac{\partial \mathcal{H}}{\partial a^\mu_i} + D_j P^{ji}_\mu = 0. \tag{3.10b}$$

[6] When the Legendre map ϕ_L is only a local diffeomorphism, the same construction holds locally.

Since $\frac{\partial \mathcal{H}}{\partial P^{ij}_{\mu}} = F^{\mu}_{ij}$, equations (3.10*a*) yield the kinematic admissibility of the critical sections γ, while equations (3.10*b*) are the translation in Hamiltonian terms of the field equations (3.5*b*).

It is worth noting that the Poincaré–Cartan representation (3.4) of the field equations is especially useful in the study of symmetries and conserved quantities. In this respect, it is easily seen that all the results stated in [1] apply equally well to the present gauge-invariant approach.

References

[1] Cianci R, Vignolo S and Bruno D 2003 The geometrical framework for Yang–Mills theories *J. Phys. A: Math. Gen.* **36** 8341–58

[2] Raiteri M 1999 Dual Lagrangians and conserved quantities in relativistic field theories *PhD Thesis* Università di Torino

[3] Ferraris M, Francaviglia M and Raiteri M 2000 Dual Lagrangian field theories *J. Math. Phys.* **41** 1889–915

[4] Saunders D J 1989 *The Geometry of Jet Bundles (London Mathematical Society, Lecture Note Series* vol 142 (Cambridge: Cambridge University Press)